Exercise sheet #8

Problem 1. We have $5 \cdot 10^{16}$ doubly charged positive ions per m^3, all moving west with a speed of 10^5 m/s. In the same region there are 10^{17} electrons per m^3 moving northeast with a speed of 10^6 m/s. (Don't ask how we managed it!) What are the magnitude and direction of $\mathbf{J}$?

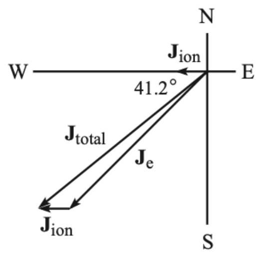
Solution: The current density is given by $\mathbf{J} = qN\mathbf{u}$. So the magnitudes of the two $\mathbf{J}$'s are

$$\begin{aligned} J_{\text{ion}} &= (2e)N_{\text{ion}}\, u_{\text{ion}} = \left(2 \cdot 1.6 \cdot 10^{-19}\mathrm{C}\right)\left(5 \cdot 10^{16}\ \mathrm{m}^{-3}\right)\left(10^5\ \mathrm{m/s}\right) = 1.6 \cdot 10^3 \frac{\mathrm{A}}{\mathrm{m}^2} \\ J_{\mathrm{e}} &= eN_{\mathrm{e}}u_{\mathrm{e}} = \left(1.6 \cdot 10^{-19}\mathrm{C}\right)\left(10^{17}\ \mathrm{m}^{-3}\right)\left(10^6\ \mathrm{m/s}\right) = 1.6 \cdot 10^4 \frac{\mathrm{A}}{\mathrm{m}^2} \end{aligned}$$

$\mathbf{J}_{\text{ion}}$ points west, and $\mathbf{J}_{\mathrm{e}}$ points southwest due to the negative charge of the electron; see Fig. below. The components of the total current density $\mathbf{J}$ are therefore

$$\begin{aligned} \mathbf{J}_{\text{west}} &= J_{\text{ion}} + \frac{J_{\mathrm{e}}}{\sqrt{2}} = \left(1.6 \cdot 10^3 + \frac{1.6 \cdot 10^4}{\sqrt{2}}\right)\frac{\mathrm{A}}{\mathrm{m}^2} = 1.29 \cdot 10^4 \frac{\mathrm{A}}{\mathrm{m}^2}, \\ \mathbf{J}_{\text{south}} &= \frac{J_{\mathrm{e}}}{\sqrt{2}} = 1.13 \cdot 10^4 \frac{\mathrm{A}}{\mathrm{m}^2} \end{aligned}$$

Since $\tan^{-1}(1.13/1.29) = 41.2^\circ$, we see that $\mathbf{J}$ points in a direction 41.2° south of west. The magnitude of $\mathbf{J}$ is $\sqrt{1.29^2 + 1.13^2} \cdot 10^4\ \mathrm{A/m^2} = 1.71 \cdot 10^4\ \mathrm{A/m^2}$.



□

Problem 2. What is the potential difference between points a and b in the circuit shown in the figure below?

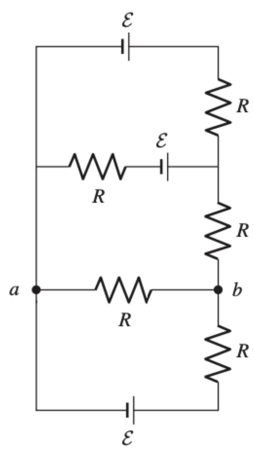
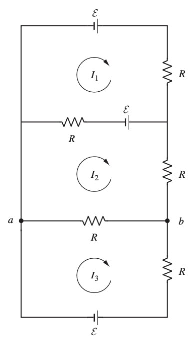


Figure 1: Caption

Solution: With the loop currents shown in the figure below the three clockwise loop equations are



$$
\begin{aligned}
0 &= \mathcal{E} - I_1 R - \mathcal{E} - (I_1 - I_2) R \\
0 &= \mathcal{E} - I_2 R - (I_2 - I_3) R - (I_2 - I_1) R \\
0 &= -\mathcal{E} - (I_3 - I_2) R - I_3 R
\end{aligned}
$$

These equations simplify to

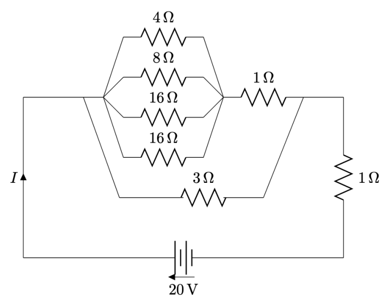
$$
\begin{aligned}
0 &= -2I_1 + I_2 \\
0 &= \mathcal{E}/R - 3I_2 + I_3 + I_1, \\
0 &= -\mathcal{E}/R - 2I_3 + I_2
\end{aligned}
$$

The third equation plus twice the second gets rid of the I_3 terms: $0 = \mathcal{E}/R - 5I_2 + 2I_1$. Adding this to the first equation then gives $0 = \mathcal{E}/R - 4I_2$. So $I_2 = \mathcal{E}/4R$, from which we quickly obtain $I_1 = \mathcal{E}/8R$ and $I_3 = -3\mathcal{E}/8R$. The potential difference between points a and b is then

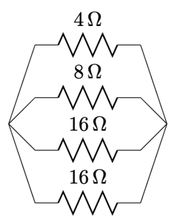
$$V_b - V_a = (I_2 - I_3) R = (\mathcal{E}/4R - (-3\mathcal{E}/8R))R = 5\mathcal{E}/8$$

This is positive, so b is at the higher potential. This makes sense by looking at the orientation of all the batteries. □

Problem 3. Find the current I in the circuit shown below:



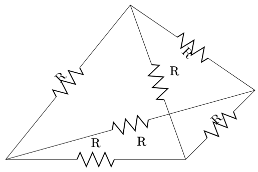
Solution: The equivalent resistance, R^1_{eq} for:



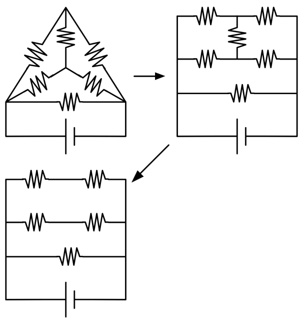
is $\frac{1}{R^1_{eq}} = \left(\frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \frac{1}{16}\right)\Omega^{-1} = \frac{1}{2}\Omega^{-1}$; therefore, $R^1_{\text{eq}} = 2\Omega$. This is in series with the 1Ω resistor. Hence, $R^2_{\text{eq}} = R^1_{\text{eq}} + 1\Omega = 3\Omega$. Now, R^2_{eq} is in parallel with the 3Ω resistance, $\frac{1}{R^3_{\text{eq}}} = \left(\frac{1}{3} + \frac{1}{3}\right)\Omega^{-1}$;

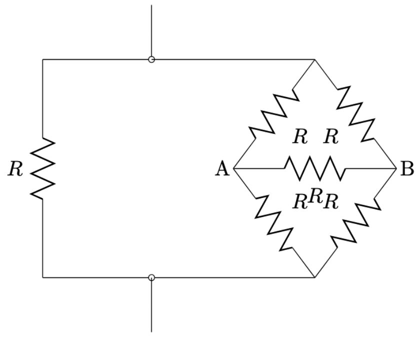
therefore, $R^3_{\text{eq}} = 1.5\Omega$ The total equivalent resistance of the circuit is $R_{\text{eq}} = (1.5+1)\Omega = 2.5\Omega$ From Ohm's law, $V = IR$, we get $I = \frac{V}{R_{\text{eq}}} = \frac{20 \text{ V}}{2.5\Omega} = 8$ A. □

Problem 4. A regular tetrahedron is a pyramid with a triangular base. Six $R = 10.0\Omega$ resistors are placed along its six edges, with junctions at its four vertices, as shown in the figure below. A 12.0-V battery is connected to any two of the vertices. Find (i) the equivalent resistance of the tetrahedron between these vertices and (ii) the current in the battery.

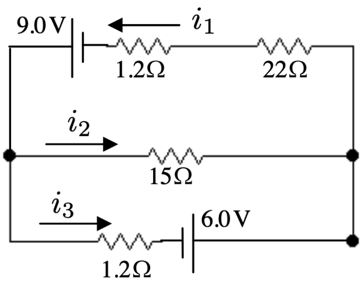


Solution. Solution: (i) First let us flatten the circuit on a 2-D plane as shown on the left in the figure below, then reorganize it to a format easier to read. Note that the voltage $V_{AB} = 0$ (figure below, right) and so the middle resistor can be removed without affecting the circuit. The remaining resistors over the three parallel branches have equivalent resistance $\frac{1}{R_{\text{tot}}} = \frac{1}{R} + \frac{1}{2R} + \frac{1}{2R} = \frac{2}{R} \Rightarrow R_{\text{eq}} = 5\Omega$. (ii) The current through the battery isn $\frac{\Delta V}{R_{\text{eq}}} = \frac{12.0 \text{ V}}{5\Omega} = 2.40$ A. □





Problem 5. Determine the magnitude and directions of the currents through $R_1 = 22\Omega$ and $R_2 = 15\Omega$ in the circuit in the figure below. Let's consider that in addition to the explicit resistances, the batteries have an internal resistance of $r = 1.2\Omega$.



Solution: here are three currents involved, and so there must be three independent equations to determine those three currents. One comes from Kirchhoff's junction rule applied to the junctionof the three branches on the left of the circuit: $i_1 = i_2 + i_3$. Another equation comes from Kirchhoff's loop rule applied to the outer loop, starting at the lower left corner, and progressing counterclockwise

$$-i_3(1.2\Omega) + 6 \text{ V} - i_1(22\Omega) - i_1(1.2\Omega) + 9 \text{ V} = 0 \Rightarrow 15 = 23.2i_1 + 1.2i_3$$

The final equation comes from Kirchhoff's loop rule applied to the bottom loop, strating at the lower left corner, and progressing counterclockwise:

$$-i_3(1.2\Omega) + 6\text{ V} + i_2(15\Omega) = 0 \Rightarrow 6 = -15i_2 + 1.2i_3.$$

Substitute $i_1 = i_2 + i_3$ into the loop equation, so that there are two equations with two unknowns

$$15 = 23.2i_1 + 1.2i_3 = 23.2\,(i_2 + i_3) + 1.2i_3 = 23.2i_2 + 24.4i_3$$

and

$$6 = -15i_2 + 1.2i_3$$

. Solve the bottom loop equation for i_2 and substitute into the loop equation, resulting in an equation with only one unknown, which can be solved

$$6 = -15i_2 + 1.2i_3 \Rightarrow i_2 = \frac{-6 + 1.2i_3}{15}$$
$$15 = 23.2i_2 + 24.4i_3 = 23.2\left(\frac{-6 + 1.2i_3}{15}\right) + 24.4i_3 \Rightarrow i_3 = 363/393.84 = 0.917\text{ A}$$
$$i_2 = \frac{-6 + 1.2i_3}{15} = -0.33\text{ A},\ \text{left.}$$
$$i_1 = i_2 + i_3 = 0.6\text{ A},\ \text{left.}$$

□